Math 1522 - Exam 1 Study Guide

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Contents

Summary and Disclaimer

This is a study guide for the first exam for math 1522 at the University of New Mexico (Calculus II). The exam covers chapter 6 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material from a calculus 1 course, including implicit differentiation. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

We begin by reviewing the methods and techniques taught in the first month of the course. We begin with invertibility of functions.

One-To-One Functions and the Horizontal Line Test

Let f be a function. Then f is called a one-to-one function if $f(x_1) \neq f(x_2)$ for any two different points x_1 and x_2 in the domain of f. Written differently, the output of f is unique for each input.
The horizontal line test is one way to show that a function is one-to-one. Draw the graph, and show that no horizontal line hits the graph at more than one point. This is often more useful for showing that a function is not one-to-one, since then you can specify a line that hits the graph at more than one point.

Once we know about one-to-one functions, we can define an inverse function. We will do this by showing how to find one.

Finding Inverse Functions

Let f be a one-to-one function. Then there is an inverse function f^{-1} which undoes f. More specifically, if $f(x) = y$, then $f^{-1}(y) = x$. To find f^{-1}, we set

$$f(y) = x$$

and solve for y on its own.
The inverse of f has the important property that $f(f^{-1})(x) = x$.

Now that we know about inverse functions, we want to know how to find the derivative of of the inverse. Fortunately, one can visualize the inverse of a one-to-one function f as flipping f over the line $y = x$. And when we flip the function over this line, the tangent line flips too. This gives us the Inverse Function Theorem:

Inverse Function Theorem

If f is a differentiable one-to-one function and $f(a) = b$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

That is, we can find the derivative of the inverse just by knowing the corresponding point on the regular function, and the derivative of the regular function. This is useful for when you need to find the derivative of a function, and you don't know how to find the inverse explicitly.

Now that we understand about inverse functions, we should try to understand exponential functions, and their inverses – called logarithms.

Exponent and Logarithm Rules

If b^x is an exponential function, then we let $\log_b(x)$ denote its inverse. We denote $\log_e(x)$ as $\ln(x)$. Because these functions are inverses, when some sort of rule holds for one, there is a similar rule that holds for the other. In this case, we have two main rules for each:

Exponent Rules

Addition Rule:	$a^x a^y = a^{x+y}$
Multiplication Rule:	$(a^x)^y = a^{xy}$

Logarithm Rules

Addition Rule:	$\log_b(xy) = \log_b(x) + \log_b(y)$
Multiplication Rule:	$\log_b(x^y) = y \log_b(x)$

Additionally, we have the change of base formula for logarithms, which is useful when we need to take derivatives and integrals:

$$\log_b(a) = \frac{\ln(a)}{\ln(b)}$$

It is worth noting that $\frac{\ln(a)}{\ln(b)}$ is not the same as $\ln(\frac{a}{b})$. This is a common mistake when first learning logarithms, and should be avoided carefully.

Now that we understand the basics about logarithms and exponents, we would like to understand how they work with calculus.

First, we will discuss their limits. And since the book doesn't mention how logarithms work in this case, you will only need to understand limits of exponents for the test.

Limits of Exponents

To evaluate limits of the form

$$\lim_{x\to\pm\infty} C^x$$

where C is a non-negative number, we have to split this up on cases depending on the size of C. This gives us:

$$\lim_{x\to\infty} C^x = 0 \text{ if } 0 \le C < 1$$
$$\lim_{x\to\infty} C^x = 1 \text{ if } C = 1$$
$$\lim_{x\to\infty} C^x = \infty \text{ if } C > 1$$

where the limit goes to positive infinity, and when the limit goes to negative infinity we have

$$\lim_{x\to-\infty} C^x = \infty \text{ if } 0 < C < 1$$
$$\lim_{x\to-\infty} C^x = 1 \text{ if } C = 1$$
$$\lim_{x\to-\infty} C^x = 0 \text{ if } C > 1.$$

Next, we would like to be able to take the derivatives of logarithms and exponential functions.

Derivatives of Exponential Functions and Logarithms

To find the derivatives of exponential functions, we only need to remember the two rules

$$(\log_b(x))' = \frac{1}{x\ln(b)} \qquad \text{and} \qquad (b^x)' = \ln(b)b^x.$$

If $b = e$, this gives us the special cases

$$(\ln(x))' = \frac{1}{x} \qquad \text{and} \qquad (e^x)' = e^x.$$

And finally, for exponential functions and logarithms, we have the integrals.

Integrals of exponential functions and $\frac{1}{x}$

Now that we know the derivatives of $\ln(x)$ and b^x, we will mention the integral of b^x and the integral of $\frac{1}{x}$.

$$\int b^x \; dx = \frac{b^x}{\ln(b)} + C \qquad \text{and} \qquad \int \frac{1}{x} \; dx = \ln|x| + C$$

We then want to know the derivatives of inverse trig functions. These are easiest to learn just by memorizing them, but you can find them all by implicit differentiation and by using trigonometric identities.

Derivatives of Inverse Trigonometric Functions

The derivatives of inverse trigonometric functions can be found with the following table:

$f(x)$	$f'(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\sec^{-1}(x)$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1}(x)$	$-\frac{1}{x\sqrt{x^2-1}}$
$\cot^{-1}(x)$	$-\frac{1}{1+x^2}$

It is also good to know about the hyperbolic trigonometric functions.

Hyperbolic Trigonometric Functions

There are two main hyperbolic trigonometric functions.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

From these, we get the hyperbolic trigonometric functions in their ”normal” way. For instance, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$. The standard derviatives have their hyperbolic forms:

$f(x)$	$f'(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\text{sech}^2(x)$
$\text{sech}(x)$	$-\text{sech}(x)\tanh(x)$
$\text{csch}(x)$	$-\text{csch}(x)\coth(x)$
$\coth(x)$	$-\text{csch}^2(x)$

Finally, we want to know L'Hopital's Rule. This tells us how to find limits of two very specific forms, namely limits which directly evaluate to be either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Unfortunately, we can't use L'Hopital's rule in other cases. However, it is still quite useful in these specialized cases.

L'Hopital's Rule

To evaluate a limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Where the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can find the derivatives. In other words:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

With all of these methods and techniques out of the way, we will now start to work out specific examples.

Worked Examples

We will now work through some examples.

Example: Evaluate

$$\lim_{x \to \infty} (\sin(x) + 2.5)^x$$

Since $\sin(x)$ is always between -1 and 1, we know that $\sin(x) + 2.5$ is always between 1.5 and 3.5. So, we can evaluate this using our standard exponential limit rules to get that

$$\lim_{x \to \infty} (\sin(x) + 2.5)^x = \infty.$$

We will now do an example with trigonometric functions in it.

Example: Find

$$\int \frac{\sec^2(x)}{1 + \tan^2(x)}\, dx$$

We begin by setting $u = \tan^2(x)$. We do this because we know that $\sec^2(x) = (\tan(x))'$, and we hope that this will cause some problems to go away. Indeed, since $du = \sec^2(x)\, dx$, we have that

$$\int \frac{\sec^2(x)}{1 + \tan^2(x)}\, dx = \int \frac{1}{1 + u^2}\, du.$$

And we know the value of this integral, since $(\tan^{-1}(x))' = \frac{1}{1+x^2}$.

$$\int \frac{\sec^2(x)}{1 + \tan^2(x)}\, dx = \tan^{-1}(u) + C.$$

However, since $u = \tan(u)$, $\tan^{-1}(u) = \tan^{-1}(\tan(x)) = x$. So,

$$\int \frac{\sec^2(x)}{1+\tan^2(x)}\ dx = x + C.$$

Indeed, one may note that $1+\tan^2(x) = \sec^2(x)$. So this result is not surprising at all.

Next up is finding the derivative of an exponential function using the chain rule.

Example: Find the derivative of $x^{\tan(x)}$.
We begin by rearranging in a clever way. We take the logarithm of this function, and raise that logarithm to the power e. In doing so, we are composing a function with its inverse, so we aren't actually changing the value of the function at all.

$$x^{\tan(x)} = e^{\ln(x^{\tan(x)})}.$$

Then, we use our logarithm rules. This gives us $\ln(x^{\tan(x)}) = \ln(x)\tan(x)$.

$$x^{\tan(x)} = e^{\tan(x)\ln(x)}.$$

Then, we use the chain rule. This gives us that

$$\left(x^{\tan(x)}\right)' = e^{\tan(x)\ln(x)}\left(\tan(x)\ln(x)\right)'.$$

Finally, we use the product rule to get that

$$\left(x^{\tan(x)}\right)' = e^{\tan(x)\ln(x)}\left(\frac{\tan(x)}{x} + \ln(x)\sec^2(x)\right).$$

Next, we find the tangent line to a curve.

Example: Let $f(x) = e^{x^2} + x$. Find the tangent line to $f(x)$ at $x = 0$.
We begin by noting that $f(0) = 1$. After this, we need to find the derivative of f. The chain rule and power rule give us that

$$f'(x) = 2x \cdot e^{x^2} + 1$$

So, $f'(0) = 1$. Thus, the tangent line has slope 1. Finally, we use the point-slope form of a line to get the equation of the line:

$$y - 1 = 1(x - 0)$$

or

$$y = x + 1.$$

These examples are all well and good, but now we want to work on a harder example. This one features logarithms, derivative rules, and the fundamental theorem of calculus for good measure. This is by far the trickiest of the examples that we will work through here.

Example: Find the second derivative of

$$\int_{\pi+e}^{x} \ln(x)e^x + 2^x \; dx.$$

By the fundamental theorem of calculus, the first derivative is just

$$\ln(x)e^x + 2^x.$$

So, we only actually need to find the derivative of this function. And we know that $(2^x)' = \ln(2)2^x$. So, the product rule gives us the other half:

$$(\ln(x)e^x + 2^x)' = \frac{1}{x}e^x + e^x \ln(x) + \ln(2)2^x.$$

Which is to say,

$$\left(\int_{\pi+e}^{x} \ln(x)e^x + 2^x \; dx\right)'' = \frac{e^x}{x} + e^x \ln(x) + \ln(2)2^x.$$

The next example will help us review one-sided limits.

Example: Find the value of

$$\lim_{x\to 0^+} \frac{x}{\ln(x)}$$

We notice that this puts us in the $\frac{\infty}{\infty}$ case of a limit, so we can use L'Hopital's rule. This gives us that

$$\lim_{x\to 0^+} \frac{x}{\ln(x)} = \lim_{x\to 0^+} \frac{1}{\frac{1}{x}}$$

or,

$$\lim_{x\to 0^+} \frac{x}{\ln(x)} = \lim_{x\to 0^+} x = 0.$$

And now, for good measure, we will do one last example. This one is not particularly hard, but is still good to have seen.

Example: Find the limit of

$$\lim_{x\to\infty} x^{-x}.$$

Since x is increasing, the value of $\frac{1}{x}$ is decreasing. However, this means that raising it to a large power will only make it smaller. So,

$$\lim_{x\to\infty} x^{-x} = 0.$$

Practice Problems

These practice problems are similar to the worked examples, but are separate from their solutions. They should be used to make sure that you are confident with the material, and are of approximately the same level of difficulty as exam questions.

1. Evaluate
$$\int \frac{(\ln(x))^{100}}{x}\,dx$$
2. Find the derivative of
$$x\log_{10}(x)$$
3. Find the value of
$$\lim_{x\to 0^+} x^2\log_3(x)$$
4. Evaluate
$$\int 2x\cdot 15^{x^2}\,dx$$
5. Evaluate
$$\int \frac{x}{x^2+1}\,dx$$
6. Evaluate
$$\int \frac{\sec^2(x)}{5+\tan(x)}\,dx$$
7. Find the value of
$$\lim_{x\to 0^+} x^3+(0.5)^{\frac{1}{x}}$$

Practice Problem Solutions

1. Evaluate

$$\int \frac{(\ln(x))^{100}}{x}\ dx$$

> **Solution:** We will use u-substitution. Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$. So,
>
> $$\int \frac{(\ln(x))^{100}}{x}\ dx = \int u^{100}\ du = \frac{u^{101}}{101} + C.$$
>
> Then, we substitute back in to x, which gives us the final answer:
>
> $$\int \frac{(\ln(x))^{100}}{x}\ dx = \frac{(\ln(x))^{101}}{101} + C.$$

2. Find the derivative of

$$x \log_{10}(x)$$

> **Solution:** We begin by using the product rule
>
> $$(x \log_{10}(x))' = (x)' \log_{10}(x) + x(\log_{10}(x))'.$$
>
> And then we recall that $x' = 1$, and that the derivative of $\log_{10}(x)$ is $\frac{1}{x\ln(10)}$. So, after simplifying, we have that
>
> $$(x \log_{10}(x))' = \log_{10}(x) + \frac{1}{\ln(10)}$$

3. Find the value of

$$\lim_{x \to 0^+} x^2 \log_3(x)$$

> **Solution:** We will use L'Hopital's rule. Specifically, we will write
>
> $$x^2 \log_3(x) = \frac{\log_3(x)}{\frac{1}{x^2}}.$$
>
> So, by L'Hopital's rule,
>
> $$\lim_{x \to 0^+} x^2 \log_3(x) = \lim_{x \to 0^+} \frac{\frac{1}{\ln(3)x}}{-2\frac{1}{x^3}}.$$
>
> Simplification gives us that this is

$$\lim_{x\to 0^+} -\frac{1}{2\ln(3)}\frac{1}{x^2}$$

And we know that $\frac{1}{x^2}$ goes to infinity as x goes to 0 from the left. So, not forgetting the negative sign, we have that

$$\lim_{x\to 0^+} x^2 \log_3(x) = -\infty.$$

4. Evaluate

$$\int 2x \cdot 15^{x^2}\ dx$$

Solution: We begin by setting $u = x^2$. Then $du = 2x\ dx$. So,

$$\int 2x \cdot 15^{x^2}\ dx = \int 15^u\ du.$$

And we know how to evaluate such an integral. So,

$$\int 2x \cdot 15^{x^2}\ dx = \frac{15^u}{\ln(15)} + C$$

Or, after switching back to x,

$$\int 2x \cdot 15^{x^2}\ dx = \frac{15^{x^2}}{\ln(15)} + C$$

5. Evaluate

$$\int \frac{x}{x^2+1}\ dx$$

Solution: We begin by setting $u = x^2 + 1$. Then $\frac{1}{2}du = x\ dx$. So,

$$\int \frac{x}{x^2+1}\ dx = \int \frac{1}{u}\ du$$

And we know how to evaluate this integral. So,

$$\int \frac{x}{x^2+1}\ dx = \ln|u| + C = \ln|x^2+1| + C$$

And since $x^2 + 1$ is always positive, this is just

$$\int \frac{x}{x^2+1}\ dx = \ln(x^2+1) + C$$

6. Evaluate

$$\int \frac{\sec^2(x)}{5+\tan(x)}\ dx$$

Solution: Set $u = 5 + \tan(x)$. Then $du = \sec^2(x)\ dx$. So,

$$\int \frac{\sec^2(x)}{5+\tan(x)}\ dx = \int \frac{1}{u}\ du.$$

And we know how to evaluate this. So,

$$\int \frac{\sec^2(x)}{5+\tan(x)}\ dx = \ln|u| + C,$$

or,

$$\int \frac{\sec^2(x)}{5+\tan(x)}\ dx = \ln|5+\tan(x)| + C.$$

7. Find the value of

$$\lim_{x\to 0^+} x^3 + (0.5)^{\frac{1}{x}}$$

Solution: We split this into two limits.

$$\lim_{x\to 0^+} x^3 + 2^{\frac{1}{x}} = \lim_{x\to 0^+} x^3 + \lim_{x\to 0^+} (0.5)^{\frac{1}{x}}.$$

We can plug $x = 0$ directly in for the first of these limits. For the second, we need to notice that as x goes to 0 from the left, we have that $\frac{1}{x}$ goes to infinity. So,

$$\lim_{x\to 0^+} x^3 + 2^{\frac{1}{x}} = 0 + \lim_{t\to\infty} (0.5)^t$$

And this exponential limit goes to zero, since $0.5 < 1$. So,

$$\lim_{x\to 0^+} x^3 + 2^{\frac{1}{x}} = 0 + 0 = 0.$$